Blast dynamics in a dissipative gas

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The blast caused by an intense explosion has been extensively studied in conservative fluids, where the Taylor-von Neumann-Sedov hydrodynamic solution is a prototypical example of self-similarity driven by conservation laws. In dissipative media however, energy conservation is violated, yet a distinctive self-similar solution appears. It hinges on the decoupling of random and coherent motion permitted by a broad class of dissipative mechanisms. This enforces a peculiar layered structure in the shock, for which we derive the full hydrodynamic solution, validated by a microscopic approach based on Molecular Dynamics simulations. We predict and evidence a succession of temporal regimes, as well as a long-time corrugation instability, also self-similar, which disrupts the blast boundary. These generic results may apply from astrophysical systems to granular gases, and invite further cross-fertilization between microscopic and hydrodynamic approaches of shockwaves.

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A blastwave follows the rapid and localized release of a large amount of energy in a medium. The physics community got seasonably interested in the dynamics of such shocks in air in the early 1940s. Taylor [1], von Neumann [2] and Sedov [3] independently understood that, as a result of the global conservation of mass and energy, the extension $R$ of the blast had to grow with time like a power law $t^\alpha$, with $\delta = 2/5$ (or $2/(d + 2)$ in dimension $d$) [4]. From a few publicly available snapshots of the blast at different times, Taylor could then estimate with an accuracy of 10% the strength of the Trinity detonation in 1945, at the time a classified information [5].

Remarkably, the hydrodynamic description of the flow inside the blast, now known as the Taylor-von Neumann-Sedov solution (TvNS), is self-similar in time, depending only on the rescaled radial distance $r/R(t)$. This similarity is of the first kind [5], i.e. driven by global invariants, and all exponents can thus be derived by dimensional analysis. This solution found widespread relevance beyond its initial realm, notably in plasma physics and astrophysics to describe laser-induced shocks [6, 7] and the evolution of supernova remnants [8]. However, it proves essential for a wealth of further applications to relax some of the conservation laws [5, 9], especially allowing for energy production or dissipation: on the shock boundary (e.g. a chemical reaction front) or in the bulk (e.g. collisional or radiative losses). This is usually expected to entail self-similarity of the second kind, where scaling exponents are no longer globally fixed, but depend continuously on parameters of the dynamics [3].

The subject of our inquiry, a blast with bulk energy dissipation, runs contrary to that expectation. Understanding its similarity properties requires connecting two levels of analysis, coarse-grained and microscopic, that have remained largely impervious to each other. Indeed, the abundant literature following the TvNS model focuses on global scaling laws or hydrodynamic models. Yet, a new arena was opened by the study of granular gases [11]: they provide prototypical dissipative media where experiments [13, 14] and particle-based simulations [15] have been performed, but a continuum view is missing. It is our purpose here to bridge this gap [16]. We provide an analytical understanding for three numerical observations: the spatial profiles for hydrodynamic fields, the scaling regimes exhibited by $R(t)$, and a previ-
ously unreported corrugation instability that distorts the shockwave at late times. Unraveling the wave structure is key in explaining these properties; Fig. 1 summarizes its main features, with a dense shell divided in three regions. The corrugation instability, also self-similar, is unique to the dissipative blast and differs significantly from those described in various other blastwaves [17, 18]. We will argue that, under broad assumptions, our results are largely independent of the mechanism at play in energy dissipation.

Model and previous results: Our model system is a granular gas of identical spherical grains with radius $\sigma$ and unit mass, where inelastic binary encounters conserve momentum but dissipate kinetic energy. Core results will not depend on a specific dissipation mechanism, hence we opt for simplicity: the energy loss is quantified by a fixed restitution coefficient $0 \leq \alpha \leq 1$ [11]; collisions are dissipative when $\alpha < 1$ and energy conserving (elastic) for $\alpha = 1$. We set a break-shot initial condition, where all grains are at rest except within a small region [13][19]. A cascade of collisions follows, with an ever growing number of particles in motion, which forms the blast as observed in Fig. 1. Its radius $R(t)$ is defined so as to contain only moving particles. Since the external medium is at rest, strong shock (infinite Mach number) conditions are ensured for any value of the initial energy release [20]. A priori, the dynamics are specified by two parameters: $\alpha$ and the volume fraction $\phi_{\text{rest}} = n_{\text{rest}} \mathcal{V}_d(\sigma)$, where $n_{\text{rest}}$ is the number density of particles in the gas at rest, and $\mathcal{V}_d(\sigma)$ is the volume of a grain in dimension $d$. In the remainder, numerical results are obtained from Molecular Dynamics (MD) simulations [11, 24].

The first property of interest is the scaling law(s) obeyed by the radius $R(t)$. As the TvNS scaling crucially hinges on energy conservation, relaxing the latter generically causes self-similarity to break down or cross over to the second kind – it is then sensitive to microscopic parameters and derivable through methods such as renormalization, but not dimensional analysis [5]. However, following an argument due to Oort [10] for astrophysical systems, a blast in a gas with bulk dissipation should tend to form a thin, hollow shell that slows down only by accreting more material. Its total radial momentum, of order $R^4 dR/dt$, is thus constant, at odds with the energy-conserving case (as explained below). This implies $R \propto t^\delta$ with $\delta = 1/(d+1)$, or $1/4$ in three dimensions – smaller than its elastic counterpart $2/(d+2)$. This solution is known as the Momentum Conserving Snowplow (MCS) [8] and is self-similar of the first kind. Recent numerical studies of model granular gases have confirmed both this scaling law and the hollow structure of the blast for any $\alpha < 1$, although the shell is comparatively thick due to the high densities considered [15].

Hydrodynamic description: Previous works on the granular blast have stopped short of investigating its spatial structure beyond these simple arguments. We now turn to a continuum description, which will shed light on the origin of the peculiar shell in Oort’s argument, and reveal its long-term instability. In order to establish a closed set of hydrodynamics equations, we define the granular temperature (or energy of random motion) of the medium through the variance of local velocity fluctuations [11, 22][24]. To account for the coupling between these temperature field $\Theta(r)$ and the density and velocity fields $n(r)$ and $u(r)$, we resort to the dense fluid transport framework for inelastic hard spheres [25], which generalizes earlier descriptions of dilute systems [22, 27].

\begin{align}
\partial_t n + \nabla(nu) &= 0 \\
\partial_t + u \nabla \cdot u + \frac{1}{n} \nabla \cdot p &= 0 \\
n(\partial_t + u \cdot \nabla) \Theta + \frac{2}{d} (p \cdot \nabla) \cdot u &= -\Lambda.
\end{align}

The energy sink term takes the form [25, 26]

$\Lambda = \omega (1 - \alpha^2) n \Theta$

and $\omega = \omega_0 n_0 \sigma^{d-1} \Theta^{1/2}$ is the local collision frequency, proportional to the average relative velocity and the inverse mean free path, with $\omega_0$ a dimensionless constant. These equations are closed by specifying the pressure tensor $p$ with a constitutive relation, discussed below, which may be taken of zeroth order in gradients, thus neglecting heat conduction [12].

In the particular elastic case ($\alpha = 1$, hence $\Lambda = 0$) with isotropic pressure $p = p I$ ($I$ being the identity matrix), the Euler equations for a perfect fluid are recovered. Within the blast, the fields then assume a scaling form

\begin{align}
n(r,t) &= n_{\text{rest}} M(\lambda), \quad u(r,t) = \frac{r}{t} V(\lambda) \\
\Theta(r,t) &= \frac{r^2}{t^2} T(\lambda), \quad p(r,t) = n_{\text{rest}} \frac{r^2}{t^2} P(\lambda)
\end{align}

where $r = r \mathbf{e}_r$ denotes the position relative to the center of the blast with $r = |r|$, and $\lambda = r/R(t)$ is the scaling variable. With the further assumption of an ideal gas constitutive relation $p = n \Theta$, Eqs. (1) and (2) together admit the classical TvNS solution [11][3][5]. It is noteworthy that there is a unique velocity scale in the elastic problem: $u$ and $\sqrt{\Theta}$ exhibit the same scaling in $R(t)/t$, meaning that coherent and incoherent motion remain coupled. This classic solution exhibits a simple structure, with a boundary layer of fixed size (the shock front) around an isotropic, self-similar blast region. The front, where discontinuities in the coarse-grained fields occur, can be defined microscopically as the thin mixed region where mobile and immobile particles coexist.

Results and discussion: In the dissipative case, the front is unchanged, but the core becomes more complex, and cannot be described by fields with a simple scaling form. Indeed, the dissipation term in Eq. (1c) depends
explicitly on an additional time scale, the collision time \( \omega^{-1} \), and is consequently incompatible with the scaling \[3\]. In the front, the speed in velocity is of order \( \dot{R} \), since particles advancing at that speed collide with others at rest. Incoherent motion is continuously generated by these collisions, so that at the boundary \( \Theta \sim R^3(t) \), as in the elastic case. Upon inserting this ansatz in Eq. (1c), the dissipation term grows in magnitude compared to transport terms by a factor \( \omega R(t)/R(t) \sim R(t) \): asymptotically, temperature is dissipated too fast to be advected on distances comparable to \( R(t) \). Dissipation instead acts over a distance \( R/\omega \) that turns out to be time independent; this creates a traveling wave-like zone behind the front, the ‘cooling region’. Then, moving further toward the interior of the blast, we anticipate a distinct cold layer where temperature is negligible and particle velocities are aligned; dissipation is thereby eliminated, and fields may once again assume a self-similar form. This region reaches a density close to random close packing, \( n_{\text{rcp}} \approx n_{\text{rest}} \rho_{\text{rcp}} \) for \( \delta = 3 \) or angular sector \( \delta = 0 \), as sketched above. The reason for the convergence \[3\] = \( \lambda^d \) (1 – \( M_{\text{rcp}}^{-1} \)) \( \lambda^{-d} \), where all three fields are parameterized by the compression \( M(x) \) obeying the following equation (which may be integrated numerically for any choice of \( Z(n) \))

\[
(M - 1) \left[ \frac{d}{2} Z^{-1} + 1 \right] = \frac{M^2}{2} - \omega_0(1 - \alpha^2) \int_0^x \frac{d(m - 1)}{2Z} \right)^{3/2}. \tag{6}
\]

Higher-order transport terms neglected in Eqs. \[1\] may in fact intervene in this layer, which has no growing typical length scale; however, this simplified analysis will prove accurate enough.

In the cold region, we can assume scaling forms similar to Eqs. \[3\], although temperature is vanishingly small. An analytic solution can be obtained assuming the fluid to be at random close packing density \( n_{\text{rcp}} = n_{\text{rest}} \rho_{\text{rcp}} \), a fair approximation corresponding to a volume fraction \( \phi_{\text{rcp}} \approx 0.84 \) for \( d = 2 \) or 0.64 for \( d = 3 \):

\[
V(\lambda) = \delta (1 - M_{\text{rcp}}^{-1}) \lambda^{-d},
\]

\[
P(\lambda) = \delta^2 \lambda^d (1 - M_{\text{rcp}}^{-1}) (M_{\text{rcp}}(\lambda^d - 1) + 1). \tag{7}
\]

The profiles thus obtained are seen in Fig. \[2\] to fare remarkably against MD simulations, despite the piecewise nature of the model. No fitted parameter is necessary. This is a crisp validation of the hydrodynamic view developed here as, to the best of our knowledge, no other model for blasts (dissipative or conservative) has been successfully applied to a dense fluid or supported by microscopic simulations. As anticipated, the flow velocity is maximal on the inner boundary of the cold region: the shell is pushed outward by its innermost particles, while the outermost slow down with dissipation and accrete onto the incoming “snowplow”. At odds with the conservative case, coherent flow thus decouples from thermal agitation; we now see that this is the cornerstone of the similarity solution.

From the tensorial form of pressure, Eq. \[1\], it can be readily shown by integrating Eq. \[1\] that the radial momentum \( \Pi = \int n u r^{d-1} dr \) within a given solid angle \( (d = 3) \) or angular sector \( (d = 2) \), is conserved. This formally demonstrates the invariant suggested by previous authors \[10\] \[15\] which yields the growth exponent \( \delta = 1/(d+1) \), as sketched above. The reason for the conservation of \( \Pi \) is twofold: first, the central pressure at \( \tau = 0 \) vanishes in the cavity, and second, there are no orthoradial exchanges of momentum, due to the decoupling of coherent (radial) and incoherent (partly orthoradial) motion. In the elastic case however \( \alpha = 1 \), the opposite statements hold, and the expansion is dominated by the central pressure, which causes \( \Pi \) to increase with time, leading to \( \delta > 1/(d+1) \). On shorter timescales, the evolving interplay of these forces gives rise to a succession of intermediate regimes, such as the Pressure-Driven Snowplow \[8\], which we confirm in MD simulations \[30\].
Finally, while the previous analysis justifies the main features of the MCS solution, we now evidence an instability arising at late times, which disrupts the solution and manifests as a growing corrugation of the shell (see Fig. 2). Numerics show that this growth follows a power law, suggesting that it stems from the self-similar cold region. While other instabilities can be found in TvNS shockwaves under specific conditions [34], we stress the generic character of the instability discussed here. Necessary conditions are the conservation of radial momentum and a vanishing pressure on the inner boundary of the shell, which explain the absence of this phenomenon in conservative gases, and distinguish it from other instabilities of granular systems [25]. We perform a linear stability analysis, focusing here on the bidimensional case which lends itself to experimental confirmation. For each field \( \psi \) (among density, velocity and pressure) in the cold region, we look for a solution of the form

\[
\psi(r,t) = \psi_0(\lambda) (1 + \delta \psi(\lambda) \cos(k \theta) t^s),
\]

with \( \psi_0(\lambda) \) the unperturbed self-similar profile, \( \delta \psi(\lambda) \) the relative perturbation, and \( s(k) \) the exponent of relative growth for a given angular frequency \( k \). This analysis is complicated by the fact that the underlying solution is neither uniform nor stationary. We have to resort to a method previously used in conservative blasts [18]: the exponent \( s \) is used as a free parameter, selected for each value of \( k \) to minimize the difference between numerically integrated profiles and theoretical values on the boundaries [30]. We thereby sample the dispersion relation \( s(k) \): as seen in Fig. 3, we predict the fastest growing perturbation with \( s \approx 0.3 \). The value of the exponent and its independence on parameters \( \alpha \) and \( \phi_{rest} \) are both confirmed by simulations.

\[ R(\theta, t) \]

\[ |\delta R(t)| \]

\[ R(s) \]

\[ 0.1 \]

\[ 0.05 \]

\[ 0.02 \]

\[ t = 5 \]

\[ t = 25 \]

\[ t = 40 \]

\[ \pi \theta \]

\[ \delta + s \]

\[ \theta \]

\[ \alpha \]

\[ u(\lambda) / u_{RH} \]

\[ n(\lambda) / n_{RH} \]

\[ \Theta(\lambda) / \Theta_{RH} \]

\[ R / R(t) \]

\[ t \]

\[ k \]

\[ 0 \]

\[ 200 \]

\[ 400 \]

\[ -2 \]

\[ -1 \]

\[ 0.3 \]

\[ 0 \]

\[ 0.2 \]

\[ 0.3 \]

\[ 0.4 \]

\[ 0.5 \]

Conclusion: Most extensions to the Taylor-von Neumann-Sedov blast exhibit either a breakdown of similarity, or self-similarity of the second kind, i.e. continuous dependence on dynamical parameters [5]. The dissipative blast studied here is exceptional in that its asymptotic regime remains self-similar of the first kind: its expansion is driven by inertial motion, rather than tuned by the dissipative processes themselves. This property is generic to a large class of fluids, from granular gases to astrophysical systems: regardless of its mechanism, bulk energy dissipation only comes into play to enforce the layered structure of the shock. Under weak conditions, the energy scales for coherent and incoherent motion decouple and each comes to dominate a different region: temperature resides only in thin layer, pushed by a growing self-similar cold fluid where motion is entirely coherent. We have given a full hydrodynamic description of this structure, in excellent agreement with particle-level simulations, and thus brought to the fore the mechanisms underlying the similarity solution. Its cornerstone, the conservation of radial momentum per angular sector, stems from this decoupling of energy scales, reflected by the anisotropic pressure within the shell. Using the computed hydrodynamic profiles, we could perform a stability analysis, and successfully predict the existence and exponent of a corrugation instability rooted in the cold region. These results invite further contact between particle-based and continuum approaches, and between fields, from plasmas to granular systems, to deepen our understanding of dissipative fluids.

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![FIG. 2: Hydrodynamic profiles from MD simulations for dense conservative (crosses) and dissipative (circles) fluids with \( \phi_{rest} = 0.3 \), and analytical solutions (solid lines). Profiles are rescaled by the Rankine-Hugoniot boundary value (see [30]), from left to right: \( n(\lambda) / n_{RH} \), \( u(\lambda) / u_{RH} \) and \( \Theta(\lambda) / \Theta_{RH} \) with \( \lambda = r / R(t) \). The conservative solution is partitioned into the gas at rest, the shock front (shaded area) and the self-similar bulk. This is the first validation of a TvNS-like solution in a dense fluid. For any dissipation i.e. \( \alpha < 1 \) (here \( \alpha = 0.8 \)), the bulk structure becomes threefold, as shown by dashed vertical lines: the cooling region between the front and \( R_0 \approx 0.93 R \); the maximally dense cold fluid down to \( R_1 \approx 0.78 R \); and the central cavity.](image)

![FIG. 3: Left: Shock radius by angular sector \( R(\theta, t) \) at successive times (see [30] for its definition). Right: Corrugation width \( \delta R \sim t^{\delta + s} \) with theoretical exponent \( s \approx 0.3 \) (solid line) and numerical validation for \( \alpha = 0.3 \) (circles) and \( \alpha = 0.8 \) (crosses). Inset: Real part of the dispersion relation \( s(k) \) for the unstable mode, crossing the marginal stability line \( \Re(s) = 0 \) (dashed line), with a plateau at \( \Re(s) \approx 0.3 \).](image)
The radius $R$ of the blast depends on energy $E$, mass density (at rest) $\rho$ and time $t$. Their only dimensionless combination is $R^5 \rho/(Et^2)$, hence $R \propto t^{2/5}$, which readily generalizes to arbitrary dimension $d$. 


[12] If a solution grows self-similarly, spatial gradients are of order $1/R(t)$ and higher derivatives (as found in viscous or heat conduction-like terms) vanish faster.


[16] No particle-based description was available for any type of blast, conservative or dissipative until recently [19].


[20] The initial energy thus only sets the timescale for the subsequent evolution.

[21] We used an Event-Driven algorithm. In order to avoid inelastic collapse, collisions become elastic below a relative velocity threshold $\epsilon = 10^{-10}$ [15, 29].


[24] This granular temperature does not have a thermodynamic meaning.


[30] See Supplemental Materials at (to be inserted by editor) for the equation of state, Rankine Hugoniot conditions, intermediate scaling regimes and stability analysis.


